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Perturbation Analysis of Electromagnetic Eigenmodes in Toroidal Waveguides

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Abstract—The propagation of electromagnetic waves in a loss-free inhomogeneous hollow conducting waveguide with circular cross section and uniform plane curvature of the longitudinal axis is considered. The explicit solution of Maxwell's equations cannot be given in toroidal waveguides. For small curvature the field equations can, however, be solved by means of an analytical approximation method. In this approximation the curvature of the axis of the waveguide is considered as a disturbance of the straight circular cylinder, and the perturbed torus field is expanded in eigenfunctions of the unperturbed problem. Using the Rayleigh–Schrödinger perturbation theory eigenvalues and eigenfunctions containing first-order correction terms are derived for the full spectrum of all modes including the degenerate ones. Complicated series expansions are obtained, which are represented in closed form by means of the residue theorem.

I. Introduction

THE problem of solving Maxwell's equations in toroidal waveguides has already been treated by several authors, since the subject has many different applications. The former H_{01} hollow waveguide project gave rise to the computation of the unwanted $H_{01} \rightarrow E_{11}$ mode conversion caused by curved transmission lines [5], [13]. Nowadays, an analogous practical use of toroidal structures is found in modern optical fibers [14]. Moreover, toroidal hollow waveguides are applied as antenna feeds or as closed resonators in particle and plasma physics [2].

A rigorous solution in closed analytical form of Maxwell's equations in toroidal waveguides via a transformation into the vector Helmholtz equation and Bernoulli separation cannot be obtained, since there is no suitable coordinate system [16]. If one represents the field vectors in a Cartesian system, then the vector Helmholtz equation separates into three scalar Helmholtz equations [4]. This, however, produces great difficulties when satisfying the boundary conditions at the surface of the torus. The fundamental crux seems to be that the toroidal geometry permits no strictly TM (E) or TE (H) eigenmodes, as is the case in cylindrical coordinates. An important exception is the so-called toroidally uniform case, in which the solution is independent of the longitudinal coordinate. These uniform solutions represent standing waves with only three field components in *closed toroidal cavity resonators*. Owing to this simplification, many calculations of toroidal modes are restricted to the uniform case [11], [12]. However, in a *toroidal waveguide section*, which will be considered in this paper, fields form traveling waves and there are always modal fields with generally six field components. These fields can be determined by using either numerical methods [8] or analytical series expansions.

The following analytic approach introduces a second complex plane which allows a decoupling of Maxwell's equations. A scalar inhomogeneous wave equation for the bicomplex field strength is in this way obtained. The solution to this equation can be found using the Rayleigh–Schrödinger perturbation theory [20] via a power series expansion refering to the inverse aspect ratio $\delta = a/R$ (a being the minor and R the major torus radius—Fig. 1). It can be expected that the perturbation theory will produce good accuracy for slightly bent torus waveguides for which the inverse aspect ratio is small, $0 \leqslant \delta \ll 1$. This paper considers the full spectrum of all toroidal modes with eigenvalues and eigenfunctions including the perturbation terms of first order $O(\delta)$. Following an idea from [12], the resulting series expansions for the field strengths can be rewritten in closed form via a pole series transformation using the residue theorem, as will be shown later.

II. The Helmholtz Equation

A. The Local Toroidal Coordinate System (ρ, φ, s)

The following procedure makes it possible to compute wave propagation effects in loss-free hollow conducting waveguides with local circular cross section and piecewise-uniform curvature. The total waveguide circuitry may consist of a limited number of separate pieces, as shown in Fig. 2. The planes of curvature of these particular hollow waveguides need not coincide; thus the total combination can produce a helixlike curvature in space. As shown in Fig. 2, the curvature may also be zero ($R_3 \rightarrow \infty$). This paper deals only with wave propagation phenomena inside *one* curved part as an element of the total circuitry. The so-called local or quasi-toroidal coordinate system conforms to the metallic boundaries and reduces, in the case of infinitesimal curvature, to the common circular cylinder coordinate system. Thus the straight circular cylinder is obtained as a limiting case of the curved structure. Fig. 1 gives the relationship of the local toroidal coordinates (ρ, φ, s) with the rectangular coordinates (x, y, z). Using the transformation $\rho = a\,\xi$ and $s = R\alpha$, with ρ as quasi-radial length and s as longitudinal coordinate measured along the curved axis, one obtains

$$x = Rh\cos\alpha \qquad y = Rh\sin\alpha \qquad z = a\xi\sin\varphi$$

with the metric coefficient $h(\xi, \varphi) = 1 - \delta\xi\cos\varphi$ and the inverse aspect ratio $\delta = a/R$, where a is the minor and R the major radius of the torus; φ is the poloidal and α the

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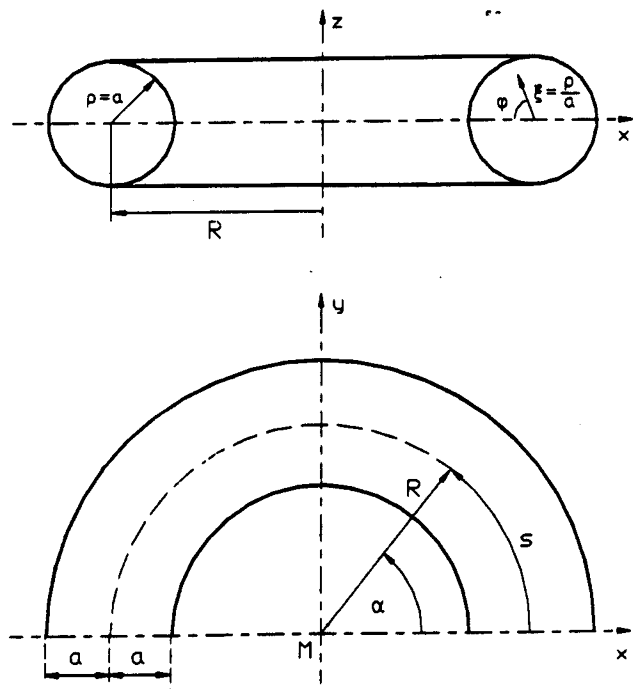

Fig. 1. Torus with coordinate systems: rectangular coordinates (x, y, z) and local toroidal coordinates (ρ, φ, s) as generalized coordinates of the straight circular cylinder. The inverse aspect ratio is $\delta = a/R$.

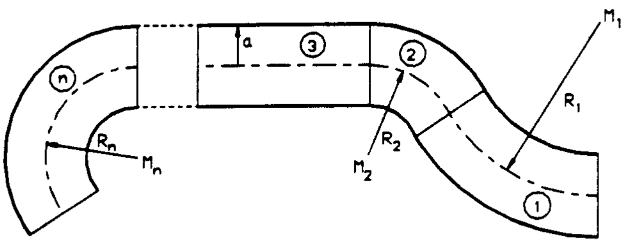

Fig. 2. Inhomogeneous waveguide circuitry consisting of n regions with circular cross section and locally constant curvature $1/R_n$.

toroidal angle. The interior of the torus is described by values of $0 \leq \xi \leq 1$.

B. The Field Equations

If we confine the considerations to uniformly curved waveguides all field components get the exponential factor $e^{j(\omega t - \beta s)}$, which is omitted in the following computations. We use $E = hE_s$ and $H = hH_s$ as abbreviation for the longitudinal field components multiplied by the metric coefficient h. Then Maxwell's equations in local toroidal components,

$$\frac{\partial E}{\xi \partial \varphi} + j\beta a E_\varphi = -j\omega\mu h a H_\rho \qquad \frac{\partial H}{\xi \partial \varphi} + j\beta a H_\varphi = j\omega\epsilon h a E_\rho$$

$$-j\beta a E_\rho - \frac{\partial E}{\partial \xi} = -j\omega\mu h a H_\varphi \qquad -j\beta a H_\rho - \frac{\partial H}{\partial \xi} = j\omega\epsilon h a E_\varphi$$

$$\frac{h\partial(\xi E_\varphi)}{\partial \xi} - \frac{h\partial E_\rho}{\partial \varphi} = -j\omega\mu\xi a H$$

$$\frac{h\partial(\xi H_\varphi)}{\partial \xi} - \frac{h\partial H_\rho}{\partial \varphi} = j\omega\epsilon\xi a E \qquad (1)$$

are transformed by elimination of the transverse fields E_ρ, E_φ, H_ρ and H_φ similar to [2], but with some change in notation. The resulting equations are thus given here in a more compact and clearer matrix-operator description:

$$(\Delta_t + \lambda)\begin{pmatrix} E \\ HZ \end{pmatrix} = \delta \begin{bmatrix} L_1 & L_2 \\ -L_2 & L_1 \end{bmatrix} \begin{pmatrix} E \\ HZ \end{pmatrix} \qquad (2)$$

where $Z = \sqrt{\mu/\epsilon}$ is the characteristic impedance in free space. Equations (2) are the required coupled longitudinal equations; their perturbational treatment plays the central role in this paper. The explicit expressions for the differential operators (Δ_t being the transverse Laplacian) are given by the following equations:

$$\Delta_t = \frac{\partial}{\xi \partial \xi}\left(\xi \frac{\partial}{\partial \xi}\right) + \frac{\partial^2}{\xi^2 \partial \varphi^2}$$

$$L_1 = \frac{1+\gamma^2}{h(1-\gamma^2)}\left(-\cos\varphi \frac{\partial}{\partial \xi} + \frac{\sin\varphi}{\xi}\frac{\partial}{\partial \varphi}\right)$$

$$L_2 = \frac{-2\gamma}{h(1-\gamma^2)}\left(\sin\varphi \frac{\partial}{\partial \xi} + \frac{\cos\varphi}{\xi}\frac{\partial}{\partial \varphi}\right) \qquad (3)$$

where γ is the dimensionless quantity $\gamma(\xi, \varphi) = \beta/(kh(\xi, \varphi))$ and $k = \omega\sqrt{\mu\epsilon}$ is the wavenumber. The coordinate dependent parameter

$$\lambda(\xi, \varphi) = (ka)^2(1 - \gamma^2(\xi, \varphi)) = (ka)^2 - (\beta a)^2/h^2(\xi, \varphi)$$

is related to the as yet unknown propagation constant β [10], which is in fact the true eigenvalue of the eigenvalue problem (2). Since β appears not only in λ but via γ also in L_1 and L_2, the toroidal waveguide defines a nonstandard eigenvalue problem [17] which is not of the typical Sturm–Liouville form [18]. A method of dealing with such unusual problems will be shown below.

The differential operator L_1 couples only modes of the same type (*EE* or *HH* coupling), and L_2 those of different type (*EH* or *HE* coupling). In the closed toroidal *cavity resonator* there exists the particular case of toroidally uniform solutions ($\partial/\partial s = 0$, i.e., $\beta = 0$), where L_2 vanishes in (3) since $\gamma = 0$. So the differential equation system (2) decouples, the *E* or *H* modes with only three field components are found. The resulting decoupled equations are identical to those published in [11]. As the coupled system (2) shows, there in contrast exist only hybrid *EH* (quasi-*E*) or *HE* (quasi-*H*) modes in the toroidal *hollow waveguide* with, in general, six nonzero field components, which are derived in the following section. Indeed, first of all the field equations are transformed to a more compact form. Generally, the coupled system of differential equations can be written as two decoupled differential equations of fourth order. To avoid their extremely cumbersome treatment a bicomplex field function F is introduced [21]:

$$F = E + iZH = h\left(E_s + i\sqrt{\frac{\mu}{\epsilon}} H_s\right). \qquad (4)$$

The function F allows a full decoupling of the field equations (2). Thus a scalar inhomogeneous Helmholtz equation

$$(\Delta_t + \lambda)F = \delta L F \qquad (5)$$

is obtained with the new differential operator $L = L_1 - iL_2$

and the boundary conditions

$$E_s|_{\xi=1}=0 \qquad \left.\frac{\partial H_s}{\partial \xi}\right|_{\xi=1}=0.$$

It should be pointed out that (5) has been deduced without any restrictive assumptions concerning the value of the inverse aspect ratio $\delta = a/R \in [0,1]$. The i complex plane thus introduced must be strictly distinguished from the j complex plane, which is commonly applied for a more elegant description of the time dependence ($\cos\omega t \to e^{j\omega t}$) using complex phasors. The following relations should be pointed out [9]: $i^2=-1$, $j^2=-1$ and the commutative product of both imaginary units is $ij = ji \neq -1$. A separation of the hypothetic field F into the physically relevant observables E and H is easily done by taking the real and imaginary parts of F in the i complex plane.

In the following section the generally valid inhomogeneous bicomplex Helmholtz equation (5) is solved by a first-order perturbational treatment valid for *small* values of delta with $0 \leqslant \delta \ll 1$.

III. Perturbation Computation of Eigenvalues and Eigenfunctions

The basic idea in solving the inhomogeneous Helmholtz equation (5) is to regard the terms caused by curvature as a perturbation of the equation for a hollow conducting waveguide with straight axis. For small values of δ the eigenvalues and eigenfunctions in the torus must proceed continuously from the solutions of the unperturbed differential equation ($\delta=0$) with increasing perturbation ($\delta>0$).

A. The Homogeneous Helmholtz Equation

In the straight circular cylinder ($\delta=0$) the Helmholtz equation (5) is reduced to

$$\left(\Delta_t + \lambda^{(0)}\right)F^{(0)} = 0. \tag{6}$$

This eigenvalue problem has a well-known solution. The solution is divided into E_{mn} and H_{mn} eigenmodes, each with five field components. Combining all double indices to one ($mn \mathrel{\hat=} \nu$ or $pq \mathrel{\hat=} \mu$) and with $\lambda_\nu^{(0)} = \tau_\nu^2$, (6) delivers the unperturbed eigenfunctions

$$F_\nu^{(0)} = \frac{1}{N_\nu} J_m(\tau_\nu \xi)\Phi_m(\varphi). \tag{7}$$

The corresponding eigenvalues $\tau_\nu = \tau_{mn} = j_{mn}$ or j'_{mn} are zeros of the Bessel function of the first kind, $J_m(j_{mn})=0$, or its derivative, $J'_m(j'_{mn})=0$. The eigenfunctions build a complete orthonormal set. The condition

$$\int_{\xi=0}^{1}\int_{\varphi=0}^{2\pi} F_\nu^{(0)} F_\mu^{(0)*}\, d\varphi\, \xi\, d\xi = \delta_{\nu\mu} = \begin{cases} 1 & \text{for } \nu=\mu \\ 0 & \text{for } \nu\neq\mu \end{cases} \tag{8}$$

defines the normalization constant N_ν using the Kronecker delta $\delta_{\nu\mu}$, where $F_\mu^{(0)}$ must be conjugated in the i complex plane. The condition $\nu=\mu$ means in detail $m=p$ and $n=q$. The azimuthal field dependence is described by trigonometric functions:

$$\Phi_m(\varphi) = \begin{Bmatrix} \cos m\varphi \\ \sin m\varphi \end{Bmatrix} \qquad \text{with } \Phi'_m(\varphi) = \begin{Bmatrix} -\sin m\varphi \\ \cos m\varphi \end{Bmatrix} \tag{9}$$

of even or odd symmetry. The notation in (9) using braces does not indicate a vector, but rather the modal orientation connected with the upper or lower functions. Every E_{mn} and H_{mn} mode has a twofold degeneracy, since transverse field components of a given mode may be either symmetric or antisymmetric relative to the $x-y$ plane (see Fig. 1). These two solutions belong to the same eigenvalue in the straight circular cylinder. In view of their different symmetry they can be treated separately, which permits one to use the (much simpler) perturbation theory for nondegenerate modes. In the straight circular cylinder there exists an additional degeneracy, $H_{0n} - E_{1n}$, which occurs because of the eigenvalue identity $j'_{0n} = j_{1n}$. The perturbational treatment of this degenerate case will be done separately. First, the perturbation terms of all nondegenerate eigenmodes will be computed.

B. Nondegenerate Rayleigh–Schrödinger Perturbation Theory of First Order

Because of the perturbation (curvature of the axis) every cylindrical eigenmode with five field components is continuously transformed to a torus mode with six field components and the same transverse symmetry. These new toroidal eigenmodes cannot be represented in closed form but they can be approximated by a series expansion. For small perturbation ($\delta \ll 1$) the nature of the field distribution of all nondegenerate eigenmodes will be changed only very slightly. Because of the perturbation the second longitudinal component (H_s or E_s) that is missing so far will appear. Thus, hybrid modes with six field components are derived, which can be classified as quasi-E (EH) and quasi-H (HE) modes [2].

To solve (5), a linear perturbation expression for the bicomplex field function F_ν and the square of the normalized propagation constant $\eta_\nu = (\beta_\nu a)^2$ can be used:

$$F_\nu = F_\nu^{(0)} + \delta F_\nu^{(1)} + O(\delta^2)$$
$$\eta_\nu = \eta_\nu^{(0)} + \delta\eta_\nu^{(1)} + O(\delta^2). \tag{10}$$

The unperturbed solutions are $F_\nu^{(0)}$ as in (7) as E_{mn} or H_{mn} mode of the straight circular cylinder and $\eta_\nu^{(0)} = (ka)^2 - \tau_\nu^2$. With $\beta_\nu = \beta_\nu^{(0)} + \delta\beta_\nu^{(1)} + O(\delta^2)$ the perturbation of the propagation constant is easily found to be

$$\beta_\nu^{(1)} a = \frac{\eta_\nu^{(1)}}{2\sqrt{\eta_\nu^{(0)}}}. \tag{11}$$

The correction terms $F_\nu^{(1)}$ and $\eta_\nu^{(1)}$ can be determined in the following way. After substituting (10) into the inhomogeneous Helmholtz equation (5) and neglecting all terms of second order $O(\delta^2)$, one obtains a differential equation for the determination of the perturbed fields:

$$\left(\Delta_t + \tau_\nu^2\right)F_\nu^{(1)} = \left(\hat{L} + \eta_\nu^{(1)}\right)F_\nu^{(0)} \tag{12}$$

with the perturbation operator $\hat{L} = L + 2\eta_\nu^{(0)}\xi\cos\varphi$. The perturbation term $F_\nu^{(1)}$ can be expanded in terms of unperturbed eigenfunctions [20]:

$$F_\nu^{(1)} = \sum_\mu c_{\nu\mu} F_\mu^{(0)}. \tag{13}$$

By applying the orthonormal property (8), the following first-order solution is derived after some short transformations:

$$\eta_\nu = \eta_\nu^{(0)} - \delta W_{\nu\nu}$$
$$F_\nu = F_\nu^{(0)} + \delta \sum_{\mu \neq \nu} \frac{W_{\mu\nu}}{\tau_\nu^2 - \tau_\mu^2} F_\mu^{(0)}. \tag{14}$$

The coupling integrals $W_{\mu\nu}$ (or $W_{\nu\nu}$ for $\mu = \nu$) can be represented in the form of inner products

$$W_{\mu\nu} = \left(F_\mu^{(0)}, \hat{L}F_\nu^{(0)}\right) = \int_{\xi=0}^{1} \int_{\varphi=0}^{2\pi} F_\mu^{(0)*} \hat{L} F_\nu^{(0)} \, d\varphi \, \xi \, d\xi. \tag{15}$$

The summation in (14) is performed over all μ except $\mu = \nu$ (i.e., $p = m$, $q = n$), which directly follows from the normalization condition for the perturbed eigenfunction $(F_\nu, F_\nu) = 1$. A more detailed description to determine the perturbed eigenfunction solution can be found in [1], [3], or [19].

The determination of the coupling integrals $W_{\mu\nu}$ is cumbersome but feasible. It can be shown that the coupling coefficients representing the self-contribution $W_{\nu\nu} = 0$. This means that the propagation constants β_ν of all nondegenerate circular cylinder eigenmodes are not altered in first-order $O(\delta)$ within the toroidal waveguide.

C. Degenerate Rayleigh–Schrödinger Perturbation Theory of First Order

The only new idea of the degenerate perturbation theory is finding those linear combinations of the unperturbed degenerate eigenfunctions which continuously result from the perturbed eigenfunctions with decreasing perturbation. There exists an infinite set of pairs of degenerate eigenmodes. For $n = 1, \cdots, \infty$ we have with $\tau_n = j_{0n}' = j_{1n}$ the following pairs of degenerate eigenfunctions:

$$F_{H_{0n}}^{(0)} = \frac{i}{\sqrt{\pi} J_0(\tau_n)} J_0(\xi \tau_n)$$
$$F_{E_{1n}}^{(0)} = \frac{\sqrt{2}}{\sqrt{\pi} J_1'(\tau_n)} J_1(\xi \tau_n) \begin{Bmatrix} -\sin\varphi \\ \cos\varphi \end{Bmatrix}.$$

Relative to the equatorial plane of the torus (see Fig. 1) we will call the symmetrical field function ($\propto \cos\varphi$) the E_{1n}' mode and the antisymmetrical ($\propto -\sin\varphi$) the E_{1n}'' mode, respectively. For a pair of modes with fixed index n, we make a thus far unknown orthogonal substitution $F_n^{(0)}$ [20]:

$$F_n^{(0)} = b_1 F_{H_{0n}}^{(0)} + b_2 F_{E_{1n}}^{(0)}. \tag{16}$$

$F_n^{(0)}$ must be normalized (see (8)); thus a first condition for the unknown i complex constants is found:

$$|b_1|^2 + |b_2|^2 = 1. \tag{17}$$

For the wanted orthogonal substitution (16) the same perturbation expression is used as in the nondegenerate case (10). The unknowns are, again, $F_n^{(1)}$ and $\eta_n^{(1)}$ as well as the constants b_1 and b_2. For $\nu = n$, (10) to (15) analogously apply to the degenerate case as well. After a lengthy but straightforward computation [6], the perturbation of the propagation constant $\beta_n^{(1)}$ is found, which leads to

$$\beta_n a = \begin{cases} \beta_n^{(0)} a \pm \delta \dfrac{ka}{\sqrt{2}\tau_n} + O(\delta^2) & \text{for } E_{1n}'' \text{ modes} \\ \beta_n^{(0)} a + O(\delta^2) & \text{for } E_{1n}' \text{ modes} \end{cases} \tag{18}$$

with $\beta_n^{(0)} a = \sqrt{(ka)^2 - \tau_n^2}$. In addition the computations deliver the ratio $b_1/b_2 = \pm 1$ for the E_{1n}'' mode, from which, together with the condition (17), the constants b_1 and b_2 can be determined, except for an unimportant phase factor. The perturbation theory also shows that a hybrid modal expression as in (16) makes no physical sense for the E_{1n}' mode, which appears to be quasi-stable. It does not degenerately couple to the H_{0n} modes; therefore the computations of the nondegenerate perturbation theory are still valid for it. For the antisymmetrical E_{1n}'' mode, in contrast, the formalism of the degenerate perturbation theory must be used. It strongly couples (even for the weakest curvature) to the H_{0n} mode, building a degenerate hybrid pair of modes:

$$F_n^{(0)} = \frac{1}{\sqrt{2}} \left(F_{H_{0n}}^{(0)} \pm F_{E_{1n}''}^{(0)} \right). \tag{19}$$

Summarizing this section, we obtain with $F_\mu^{(0)}$ from (7) the perturbed eigenfunctions in first order $O(\delta)$ of the nth degenerate hybrid wave pair (compare with (14)):

$$F_n^{\pm} = F_n^{(0)} + \delta \sum_{\tau_\mu \neq \tau_n} \frac{\left(F_\mu^{(0)}, \hat{L}F_n^{(0)}\right)}{\tau_n^2 - \tau_\mu^2} F_\mu^{(0)}. \tag{20}$$

The degenerate eigenfunctions of the new transformed basis (19) undergo, as (18) shows, a "level splitting" for nonvanishing perturbation; thus the degeneracy is removed. Finally, a closed-form representation of the field intensities of the toroidal modes is given in the next section.

IV. The Torus Field

A. Series Representations

Starting from the series expansions for the perturbation of the eigenfunctions derived above, which are lengthy and cumbersome, a method of finding compact, closed expressions for the torus fields is outlined in this section. The double sums over all $\mu \hat{=} pq$ with $p = m \pm 1$ and $q = 1, \cdots, \infty$ in (14) and (20) can be substantially shortened by summing the inner perturbation series over all q, which can generally take the following two forms:

$$S_E = \sum_{q=1}^{\infty} \frac{j_{pq}}{\left(\tau_\nu^2 - j_{pq}^2\right)^\psi} \frac{J_p(\xi j_{pq})}{J_p'(j_{pq})}$$
$$S_H = \sum_{q=1}^{\infty} \frac{\left(j_{pq}'\right)^\chi}{\left(\tau_\nu^2 - j_{pq}'^2\right)^\psi \left(j_{pq}'^2 - p^2\right)} \frac{J_p(\xi j_{pq}')}{J_p(j_{pq}')} \tag{21}$$

with $\tau_\nu = j_{mn}$ or j_{mn}' and the exponents $\psi = 2$ or 3 and $\chi = 2$ or 4. A closed-form representation of the series expansions (21) is found by means of the residue theorem using the

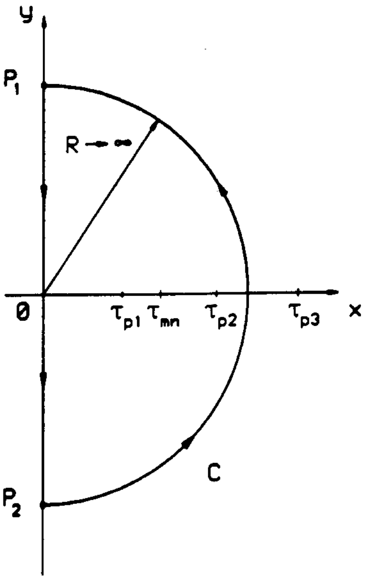

Fig. 3. Path for the residue integral for the transformation of the pole series (21).

following complex functions:

$$g_E(z) = \frac{z}{(\tau_\nu^2 - z^2)^\psi} \frac{J_p(\xi z)}{J_p(z)}$$

$$g_H(z) = \frac{z^{\chi-2}}{(\tau_\nu^2 - z^2)^\psi} \frac{J_p(\xi z)}{J_p'(z)}, \tag{22}$$

where z defines a third complex plane ($z = x + ly$ with $l^2 = -1$), different from the others ($i^2 = -1$, $j^2 = -1$) introduced earlier. Considering the closed line integral in Fig. 3 with the integrands $g_E(z)$ and $g_H(z)$ from (22), one obtains with the residue theorem of function theory [18]

$$\oint_C g_{E,H}(z)\, dz = 2\pi l \sum_k \operatorname{Res}\{g_{E,H}(z_k)\} \tag{23}$$

taken at the isolated singularities z_k. The integrands are antisymmetrical on the imaginary axis ($z = \pm ly$). Thus, the integral between P_1 and P_2 vanishes. Also, the Jordan lemma [18] shows that the integral along the half-circle makes no contribution. So, splitting the residual series (23)

$$0 = \operatorname{Res}\{g_{E,H}(z)\}_{z=\tau_\nu} + \sum_q \operatorname{Res}\{g_{E,H}(z)\}_{z=\tau_{pq}}$$

yields a closed-form summation of the unknown pole series (21). Computing the residues at the simple zeros τ_{pq} of the Bessel functions in the denominator of $g_{E,H}(z)$ (see also [11]) leads to S_E and $-S_H$; thus,

$$S_E = -\operatorname{Res}\{g_E(z)\}|_{z=\tau_\nu} \qquad S_H = \operatorname{Res}\{g_H(z)\}|_{z=\tau_\nu}.$$

Computing the residues at the poles of order ψ is straightforward but tedious and cumbersome. The closed-form expressions reached in this way for the pole series S_E and S_H from (21) replace the perturbation series (14) and (20) in the field representations, the final form of which will be given in the next section.

B. Closed-Form Expressions of First Order

After splitting the bicomplex field function (4) into real and imaginary parts in the i complex plane, the physically relevant field intensities are obtained. In the following, the longitudinal field components multiplied by the metric coefficient $h(\xi, \varphi)$ are given ($E = hE_s$, $H = hH_s$). The corresponding transverse fields can simply be deduced by some derivatives using Maxwell's equations (1).

1) EH Modes (Nondegenerate Theory): The following field representations are valid for every hybrid EH_{mn} torus mode, which results from a perturbation of a nondegenerate E_{mn} eigenmode of the straight circular cylinder, including the symmetrical E_{1n}' mode but without the antisymmetrical E_{1n}'' mode:

$$\begin{aligned} E = {} & \frac{1}{N_{mn}^E} J_m(j_{mn}\xi)\Phi_m \\ & - \frac{\delta}{2 j_{mn}^2 N_{mn}^E}\Big\{\big(2(ka)^2 - j_{mn}^2\big)\xi J_m(j_{mn}\xi)\cos\varphi\Phi_m \\ & + (\beta_{mn}^E a)^2\Big[\frac{m}{\xi} J_m(j_{mn}\xi)(1+\xi^2)\sin\varphi\Phi_m' \\ & - j_{mn} J_m'(j_{mn}\xi)(1-\xi^2)\cos\varphi\Phi_m\Big]\Big\} \end{aligned}$$

$$\begin{aligned} ZH = {} & -\frac{\delta k a \beta_{mn}^E a}{j_{mn}^2 (m^2-1) N_{mn}^E}\Big\{ j_{mn} J_m'(j_{mn}\xi) \\ & \cdot [m\cos\varphi\Phi_m' + \sin\varphi\Phi_m] \\ & + \frac{1}{\xi} J_m(j_{mn}\xi)\big[\big(\xi^2 + m^2(1-\xi^2)\big)\sin\varphi\Phi_m \\ & + m\cos\varphi\Phi_m'\big]\Big\} \end{aligned}$$

where N_{mn}^E is the normalization factor, given by

$$N_{mn}^E = J_m'(j_{mn})\sqrt{\frac{\pi}{2}(1+\delta_{m0})}.$$

Here $\beta_{mn}^E a = \sqrt{(ka)^2 - j_{mn}^2}$ is the normalized propagation constant, and the poloidal eigenfunctions Φ_m and Φ_m' are given by (9). The magnetic longitudinal field strength especially for the E_{1n}' mode with $m = 1$ results in

$$ZH = -\frac{\delta k a \beta_{1n}^E a}{2 j_{1n}^2 N_{1n}^E}\left[\frac{j_{1n}}{2} J_2(j_{1n}\xi) - \xi J_1(j_{1n}\xi)\right]\sin 2\varphi.$$

2) HE Modes (Nondegenerate Theory): The hybrid HE_{mn} torus modes, which result from a perturbation of nondegenerate H_{mn} eigenmodes of the straight circular cylinder, except the H_{0n} mode, can be obtained in a way similar to that for the EH_{mn} modes. The explicit expressions are shown in [6].

3) Degenerate Mode Pair $F_n^\pm$: Finally, the explicit expressions for the longitudinal field components multiplied by the metric coefficient $h(\xi, \varphi)$ are shown for the degenerate hybrid mode pair $F_n^\pm$ (see (20)), with which the unwelcome mode conversion $H_{01} \to E_{11}$ in circular hollow waveguide

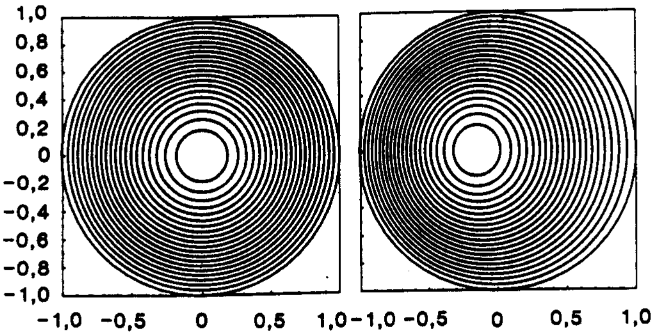

Fig. 4. Transverse magnetic field lines for the E_{01} mode in the straight circular cylinder ($\delta = 0$; *left*) and for the corresponding EH_{01} mode in the toroidal waveguide ($\delta = 0.027$; *right*).

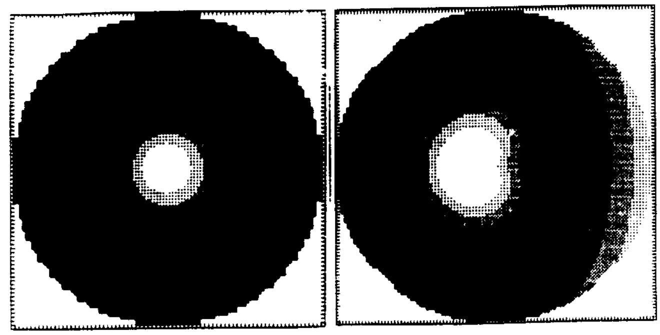

Fig. 5. The energy-flux density: transverse distribution of the longitudinal component P_s of the Poynting vector (see (24)) for the E_{01} mode in the straight circular cylinder ($\delta = 0$; *left*) and for the corresponding EH_{01} mode in the toroidal waveguide ($\delta = 0.027$; *right*).

transmission lines can be described:

$$E^{\pm} = \mp \frac{1}{N_n} J_1(\tau_n \xi)\sin(\varphi) \pm \frac{\delta}{4\tau_n^2 N_n}\Big\{(ka)^2 \xi J_1(\tau_n \xi) + \big(\beta_n^{(0)} a\big)^2\big[\tau_n(1-\xi^2)J_2(\tau_n\xi) + 3\xi J_1(\tau_n \xi)\big]\Big\}\sin(2\varphi)$$

$$ZH^{\pm} = \frac{1}{\sqrt{2}N_n} J_0(\tau_n\xi) - \frac{\delta}{4\tau_n^2 N_n}\Big\{(ka)^2\sqrt{2}\,\xi J_1'(\tau_n\xi)\cos\varphi + \big(\beta_n^{(0)}a\big)^2\sqrt{2}\,\big[\tau_n(1-\xi^2)J_1(\tau_n\xi) + \xi J_1'(\tau_n\xi)\big]\cos\varphi \pm ka\beta_n^{(0)}a\big[2\xi J_1(\tau_n\xi) - \tau_n J_2(\tau_n\xi)\big]\cos 2\varphi\Big\}$$

with $\tau_n = j_{0n}' = j_{1n}$, the normalization constant $N_n = \sqrt{\pi} J_1'(\tau_n)$, and the normalized propagation constant

$$\beta_n^{(0)} a = \sqrt{(ka)^2 - \tau_n^2}.$$

V. Field Concentration and Energy Shift

For a better physical understanding of the wave propagation phenomena in toroidal hollow conducting waveguides, plots of the perturbed field distribution are compared with those for the straight circular cylinder. The magnetic field lines in a transverse section ($s = \text{const.}$, $ka = 6$) are shown in Fig. 4 for the E_{01} mode, whereas the transverse distribution of the longitudinal component of the Poynting vector,

$$P_s(\xi, \varphi) = \frac{1}{2}\operatorname{Re}\{\vec{E}_t \times \vec{H}_t^*\}\cdot \vec{e}_s \qquad (24)$$

is displayed in Fig. 5 with the transverse field vectors $\vec{E}_t = (E_\rho, E_\varphi)$ and $\vec{H}_t = (H_\rho, H_\varphi)$. To draw the field lines, an algorithm from [7] was applied. The intensity in the energy flux increases from lighter to darker shade with linear quantization. Both diagrams indicate a considerable displacement of the field lines as well as of the energy flux toward the outer boundary of the waveguide away from the center of curvature (located at the right-hand side of the images) as the curvature increases. There appears a remarkable enhancement of the time-averaged energy-flux density P_s in the outer cross-sectional domain, while in the inner an evident reduction can be seen. This leads to an unsymmetrical energy-flux distribution in the toroidal waveguide. Corresponding diagrams are shown in Figs. 6 and 7 for the H_{11}' mode. It

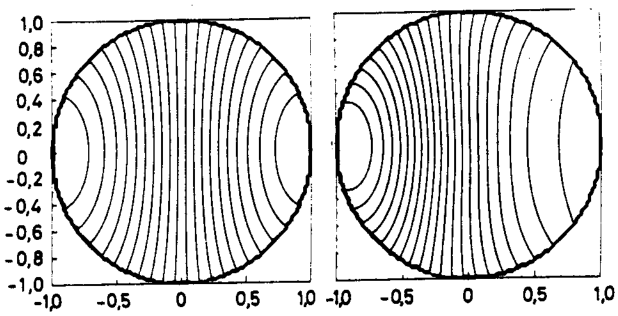

Fig. 6. Transverse electric field lines for the symmetrical H_{11}' mode in the straight circular cylinder ($\delta = 0$; *left*) and for the corresponding HE_{11}' mode in the toroidal waveguide ($\delta = 0.02$; *right*).

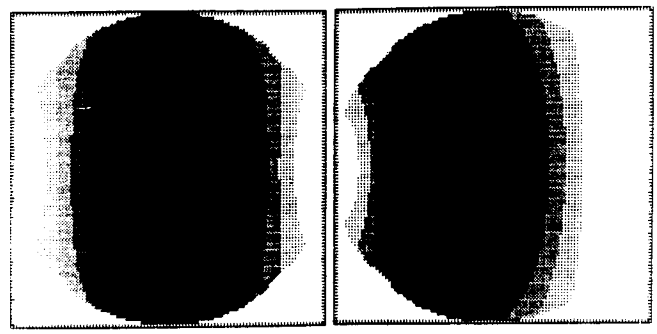

Fig. 7. The energy-flux density: transverse distribution of the longitudinal component P_s of the Poynting vector (see (24)) for the symmetrical H_{11}' mode in the straight circular cylinder ($\delta = 0$; *left*) and for the corresponding HE_{11}' mode in the toroidal waveguide ($\delta = 0.02$; *right*).

is the lowest mode in the homogeneous circular waveguide and shows a similar field shift behavior compared with the E_{01} mode. Finally, the degenerate E_{11}'' mode is shown in Figs. 8 and 9.

The lowest five circular waveguide modes ($H_{11}, E_{01}, H_{21}, H_{01}, E_{11}$) have been examined with respect to their perturbed behavior in the toroidal waveguide with the same outward-directed displacement effect always observed [6]. This agrees with the results in [15], where a field and energy shift toward the outward direction of bending was also observed for these five modes, although in [15] the possibility of an inward-directed shift for other modes (e.g., H_{51}, H_{61}) was mentioned.

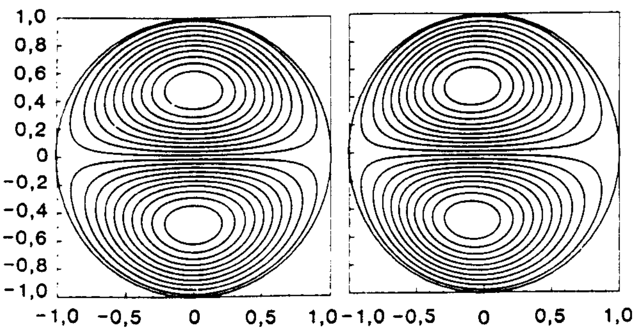

Fig. 8. Transverse magnetic field lines for the antisymmetrical E_{11}'' mode in the straight circular cylinder ($\delta = 0$; *left*) and for the corresponding EH_{11}'' mode in the toroidal waveguide ($\delta = 0.042$; *right*).

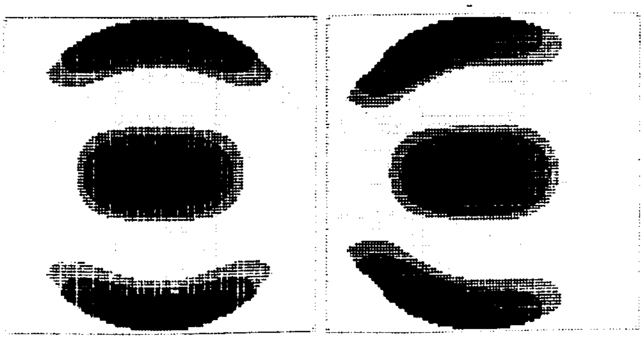

Fig. 9. The energy-flux density: transverse distribution of the longitudinal component P_s of the Poynting vector (see (24)) for the antisymmetrical E_{11}'' mode in the straight circular cylinder ($\delta = 0$; *left*) and for the corresponding EH_{11}'' mode in the toroidal waveguide ($\delta = 0.042$; *right*).

VI. Conclusion

Toroidal waveguides are applied as transitions in hollow waveguide circuitries, as feed lines of microwave antennas, and as closed resonators in accelerators and fusion power plants. A problem in which toroidal waveguides are regarded as a case of perturbed circular cylinders has been treated. The influence of a weakly curved longitudinal axis has been studied in detail. The slightly curved longitudinal axis permits an approximate solution of the field equations starting from the well-known eigenmodes of the straight circular cylinder. With a first-order perturbational approach, field correction terms have been derived in a closed-form representation. The theory developed can be further extended to certain other geometrical distortions of the cylindrical symmetry. Nonuniform curvature, serpentine bends (see Fig. 2), and torsion may be of interest. In addition the approach presented here opens the way to the more general problem of inhomogeneous waveguides with impedance boundaries or with plasma filling.

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