The Toroidal Hollow Waveguide with Smoothly Tapered Cross Section

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Abstract

The stationary electromagnetic boundary value problem with harmonic time dependence $e^{j\omega t}$ is treated for loss-free, slightly inhomogeneous hollow waveguides with circular cross section. Starting from the canonical problem of the homogeneous circular cylinder the influence of small changes of the cross sectional radius and of a slightly bent longitudinal axis on the stationary field problem is investigated. Because the shape of the hollow waveguide changes only slowly, approximate analytical solutions of the field equations can be found applying an improved adiabatic mode theory and a perturbational approach of Rayleigh and Schrödinger.

1. Introduction

Hollow waveguides with circular cross section are important microwave devices for radar-, communication- and satellite-applications. Besides the common circular cylinder some other waveguide designs are used (see Figure 3).

- circular cylinder hollow waveguide
- tapered hollow waveguide
- toroidal hollow waveguide
- toroidal tapered hollow waveguide

Hollow waveguide tapers are applied as matching devices connecting two waveguides of different diameter. An unsteady discontinuous metallic boundary would effect severe reflection- and conversion losses for the incident mode, which carries the information signal. Therefore usual taper profiles are constructed as smooth and weakly inhomogeneous transitions. Toroidal hollow waveguides with curved longitudinal axis have several applications, e.g. as antenna feed lines. With uniform plane curvature closed torus structures are used as cavity resonators in plasma physics or as toroidal antennas (*Lileg et al* [1]). Combining the tapered transition with the toroidal structure leads to a toroidal taper with weakly changing cross sectional radius and uniformly bent longitudinal axis. The exact solution of Maxwell's equations in such inhomogeneous non-canonical waveguides cannot be given with correct boundary conditions

$$\vec{n} \times \vec{E} = 0 \qquad \vec{n} \cdot \vec{H} = 0 \quad . \tag{1}$$

Deforming the boundaries of the waveguide gives rise to a disturbance of wave propagation. For perturbations, which are not too large, approximate analytical solutions of Maxwell's equations can be found. Starting from the well known eigenmode solutions of the canonical homogeneous straight circular cylinder waveguide of constant cross section all modal coupling effects are considered via suitable correction terms to the unperturbed solution.

2. The tapered hollow waveguide

A simple approach to field calculations in the tapered waveguide is found using adiabatic approximation methods. Adiabatic modes are defined as local eigenmodes of the inhomogeneous structure and adapt continuously to the changing environment (*Arnold* and *Felsen* [2]). Therefore the mutual coupling between adiabatic modes is small. The metallic boundary of the investigated tapered hollow waveguide must be smooth and only weakly changing. The usual adiabatic modes are modified in this paper to get a more precise description of wave propagation. Thus a better satisfaction of the Helmholtz equation and the boundary conditions can be obtained (*Kark* [3]).

2.1 Modified adiabatic modes - MAM

The eigenfunction solution for the vector potential in the homogeneous circular cylinder

$$A^{(0)} = C \, J_m(K\rho) \, e^{\pm j m \varphi} \, e^{\pm j k_z z} \tag{2}$$

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built up by Bessel's and trigonometric or exponential functions must be modified with appropriate correction terms taking the perturbed geometry into account. A partial separation ansatz for the vector potential $\vec{A} = A\,\vec{e}_z$ in cylindrical coordinates

$$A = f_1(\rho, z)\, f_2(\varphi)\, f_3(z) \tag{3}$$

is introduced into the Helmholtz equation with Δ as Laplacian operator

$$\left(\Delta + k^2\right) A = 0 \quad . \tag{4}$$

This leads to the azimuthal solution

$$f_2(\varphi) = \begin{Bmatrix} \cos m\varphi \\ \sin m\varphi \end{Bmatrix} \tag{5}$$

and delivers an involved partial differential equation for the quasi-radial eigenfunction $f_1(\rho, z)$

$$\begin{aligned} &\left(\frac{\partial^2}{\partial \rho^2} + \frac{\partial}{\rho\,\partial \rho} + K^2(z) - \frac{m^2}{\rho^2}\right) f_1(\rho, z) = \\ &= -2\,\frac{\partial f_1(\rho, z)}{\partial z}\,\frac{d(\ln f_3(z))}{dz} \quad . \end{aligned} \tag{6}$$

The propagation term $f_3(z)$, which incorporates distributed modal reflections, must satisfy

$$\left(\frac{d^2}{dz^2} + k_z^2(z)\right) f_3(z) = 0 \quad . \tag{7}$$

Following separation equation holds

$$k^2 = K^2(z) + k_z^2(z) \quad . \tag{8}$$

The transversal eigenvalue K and the propagation constant k_z are weakly depending functions of the longitudinal coordinate z (see eq. (13)). Eq. (7) can be solved approximately using standard WKBJ methods (*Felsen* and *Marcuvitz* [4]). The failure of the conventional adiabatic mode theory at cutoff transitions can be removed using the uniform Airy function representation of *Langer* [5]. An approximate solution of eq. (6) is found applying a Picard iterative scheme with starting value

$$f_1^{(0)} = J_m(K(z)\,\rho) \quad . \tag{9}$$

The first iteration gives $f_1^{(1)}$, which already includes all correction terms of first order $O(a')$ with $a' = da\,/\,dz$ (see eq. (13)). Q and P are suitable abbreviations.

$$f_1^{(1)} = \exp\left[Q\,\frac{dK}{K^3\,dz}\,P\right] J_m(K\rho) \tag{10}$$

$$Q(z) = \frac{d(\ln f_3)}{dz} \tag{11}$$

$$P(\rho, z) = m - \frac{(K\rho)^2}{2} - K\rho\,\frac{d(\ln J_m(K\rho))}{d(K\rho)} \tag{12}$$

The transversal eigenvalue K(z) is related via the cross sectional radius a(z) to the n^{th} zero of Bessel's function J_m (or its derivative J_m') of order m

$$K(z)\,a(z) = \begin{cases} j_{mn} & \textit{for TH-modes} \\ j_{mn}' & \textit{for TE-modes} \end{cases} \quad , \tag{13}$$

which results from the boundary conditions

$$A^{(0)} = 0\Big|_{\rho=a} \qquad \textit{for TH-modes} \tag{14}$$

$$\frac{\partial A^{(0)}}{\partial \rho} = 0\Bigg|_{\rho=a} \qquad \textit{for TE-modes} \quad . \tag{15}$$

Combining a TH- with a TE-mode leads to a hybrid coupled mode pair, which enables a better satisfaction of the boundary conditions eq. (1). The method of modified adiabatic modes was checked by *Kark* [6] considering the case of a conical linearly tapered waveguide transition. The eigenfunctions of a conical horn can exactly be given using spherical harmonics. A comparison of the conventional adiabatic mode solution (AM) using the simple radial eigenfunction $f_i^{(0)}$ with the improved expression $f_i^{(1)}$ of the modified adiabatic mode theory (MAM) indicates a 50 % reduction of the deviation from the exact spherical wave solution. Thus MAM-theory delivers a stronger tool than the older AM-theory for analyzing inhomogeneous waveguide transitions.

3. The toroidal hollow waveguide

Besides the tapered waveguide a smoothly and uniformly bent toroidal hollow waveguide is investigated. It results from a homogeneous circular cylinder waveguide, which axis is not straight but weakly curved. This symmetry deformation is considered as a perturbation of the homogeneous arrangement. With the help of Rayleigh-Schrödinger perturbation theory (*Schrödinger* [7]) suitable correction terms to the unperturbed field solutions are computed. Toroidal structures always lead to non-separable differential equations with exception of the Laplacian equation for the static case. This is the most important reason for the appearing difficulties. Thus for an analytical solution only approximation methods can be used.

The perturbation formalism of Rayleigh and Schrödinger is applied to toroidal hollow waveguides by *Kark* [8]. From Maxwell's equations a pair of coupled differential equations for the longitudinal field components is derived, which can be decoupled by the help of a bicomplex transformation (see eq. (16)). Thus an essential simplification can be achieved. In contrast to the toroidal *resonator* there exist no toroidally uniform modes with three field components in the toroidal *waveguide*. There are only hybrid fields with six components respectively. Using perturbation theory of first order all eigenfunctions and eigenvalues (i.e. the propagation constants) of the hybrid quasi-E- and quasi-H-modes can be found (*Kark* [9]). The infinite perturbation series can be summed in closed form using the residue theorem. In this paper some instructive plots of the transversal field lines and the spatial distribution of the energy flux density are given in comparison with the modal behaviour in the straight circular cylinder (see Figures 1 and 2).

3.1 The perturbed Helmholtz equation

Maxwell's equations in local toroidal coordinates (ξ, φ, α) (see Figure 3 with $\xi = \rho / a$) may be reduced in the case of uniform plane curvature to a coupled set of partial differential equations for the longitudinal field components E_α and H_α. Using a bicomplex transformation ($I^2 = -1$)

$$F = h\left(E_\alpha + I\sqrt{\frac{\mu}{\varepsilon}}\, H_\alpha \right) \tag{16}$$

$$h(\xi, \varphi) = 1 - \delta\, \xi \, \cos\varphi \tag{17}$$

with the metric coefficient h and the inverse aspect ratio δ

$$\delta = \frac{a}{R} \quad , \tag{18}$$

where a is the minor and R the major radius of the toroidal waveguide, a perturbed Helmholtz equation with suitable boundary conditions is obtained (*Kark* [10]).

$$(\Delta_t + \lambda) F = \delta \, L \, F \tag{19}$$

$$E_\alpha \big|_{\xi=1} = 0 \qquad \left. \frac{\partial H_\alpha}{\partial \xi} \right|_{\xi=1} = 0 \tag{20}$$

Δ_t is the transversal Laplacian operator and $L = L_1 - I\, L_2$ an involved perturbation operator with

$$\Delta_t = \frac{\partial}{\xi\, \partial \xi}\left(\xi \frac{\partial}{\partial \xi} \right) + \frac{\partial^2}{\xi^2\, \partial \varphi^2} \tag{21}$$

$$L_1 = \frac{1 + \gamma^2}{h\,(1 - \gamma^2)} \left(\frac{\sin\varphi}{\xi} \frac{\partial}{\partial \varphi} - \cos\varphi \frac{\partial}{\partial \xi} \right) \tag{22}$$

$$L_2 = \frac{-2\gamma}{h\left(1-\gamma^2\right)}\left(\sin\varphi\,\frac{\partial}{\partial\xi} + \frac{\cos\varphi}{\xi}\,\frac{\partial}{\partial\varphi}\right) \tag{23}$$

using the dimensionless quantity γ, depending via h on the transversal coordinates ξ and φ.

$$\gamma = \frac{\beta}{\omega\sqrt{\mu\varepsilon}\;h(\xi,\varphi)} \tag{24}$$

$$\lambda = \omega^2\mu\varepsilon\,a^2\left(1-\gamma^2\right) \tag{25}$$

The parameter λ is related via γ to the so far unknown propagation constant β, which serves as an eigenvalue of eq. (19). The i-complex plane, introduced in eq. (16), must strictly be separated from the j-complex plane, which is commonly applied for a more elegant description of the time dependence ($\cos\omega t \rightarrow e^{j\omega t}$) using complex phasors.

3.2 Perturbation theory

The basic idea to solve the non-separable Helmholtz equation (19) is to understand the curvature as a disturbance of the hollow waveguide with straight axis. The eigenvalues and eigenfunctions in the torus must continuously result from the solutions of the unperturbed differential equation ($\delta = 0$) while increasing the disturbance ($\delta > 0$) . Thus the perturbed eigenfunctions can be represented in a power series expansion refering to the inverse aspect ratio $\delta = a/R$. The expansion coefficients are linear combinations of the unperturbed eigenfunctions of the straight circular cylinder. Only toroidal waveguides with weak curvature ($0 \leq \delta \ll 1$) are considered in this paper. So the wanted expansions may be truncated after the linear term δ and a perturbation ansatz of first order for the bicomplex field function F_ν and the propagation constant β_ν is made

$$F_\nu = F_\nu^{(0)} + \delta\,F_\nu^{(1)} \tag{26}$$

$$\beta_\nu = \beta_\nu^{(0)} + \delta\,\beta_\nu^{(1)} \quad , \tag{27}$$

where all modal double indices (m,n) are combined to one (ν). The perturbation term is expanded in unperturbed eigenfunctions with so far unknown expansion coefficients $c_{\nu\mu}$

$$F_\nu^{(1)} = \sum_\mu c_{\nu\mu}\,F_\mu^{(0)} \quad . \tag{28}$$

Inserting all this in eq. (19), while neglecting all terms of second order $O(\delta^2)$, infinite perturbation series are derived, which can be represented in closed form using the residue theorem. For the resulting perturbed expressions see *Kark* [6].

3.3 Field shift

For a better physical understanding of wave propagation phenomena in toroidal waveguides some plots of the obtained field intensities compared with the field of the straight circular cylinder are finally given. The magnetic field lines in a transversal cross section are shown in Figure 1 for the TH_{11} -mode, while the transversal distribution of the longitudinal component of the Poynting vector

$$P_\alpha = \frac{1}{2}\,Re\left\{\vec{E}\times\vec{H}^*\right\}\cdot\vec{e}_\alpha \tag{29}$$

is displayed in Figure 2. Darker shade indicates higher flux. Both diagrams show a considerable shift of field lines and energy flux away from the center of curvature (located at the right hand side) towards the *outer* boundary of the waveguide as the curvature increases.

4. The toroidal taper

As generalization a combined structure is finally treated, which incorporates both distortions of circular cylindrical symmetry. The toroidal taper has a uniformly and weakly bent longitudinal axis and a slowly increasing or decreasing circular cross section. Such an inhomogeneous hollow waveguide has not yet been treated with analytical eigenfunction methods. After having investigated both symmetry distortions separately a linear superposition of the field solutions for the tapered and the toroidal waveguide leads to an approximate description of modal behaviour in the new type of waveguide (*Kark* [6]). For first order computations the correction terms coming from changing diameter $O(a')$ and those excited by the curvature $O(\delta)$ may be computed separately and may finally be added without a hybrid coupling term, which is in deed of second order $O(a'^2, a'', \delta^2, a'\delta)$ and will be neglected, provided that the geometrical perturbation stays small.

5. Conclusion

Some types of weakly inhomogeneous hollow waveguides have been investigated using analytical quasi-eigenfunction methods. As generalization of the present theory some other deformations of the cylindrical symmetry could be taken into account. One could consider non-uniform curvature, waving bends or torsion of the longitudinal axis. Also an investigation of boundaries with losses or of a plasma-filled inhomogeneous waveguide would be of special interest.

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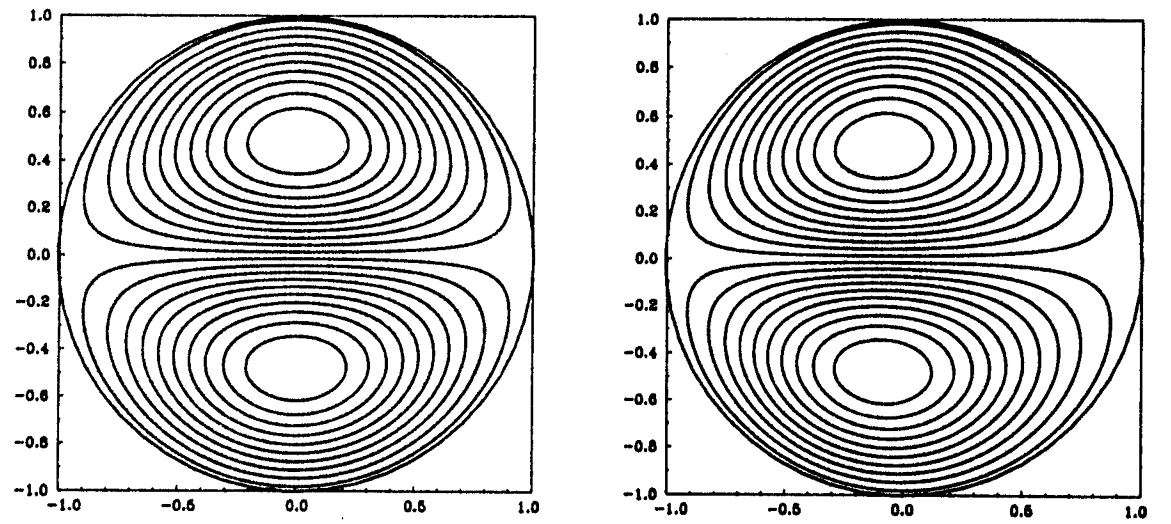

Figure 1. Magnetic field lines for the TH_{11}-mode in the straight circular cylinder ($\delta = 0$) and its perturbed behaviour in the toroidal waveguide ($\delta = 0.042$).

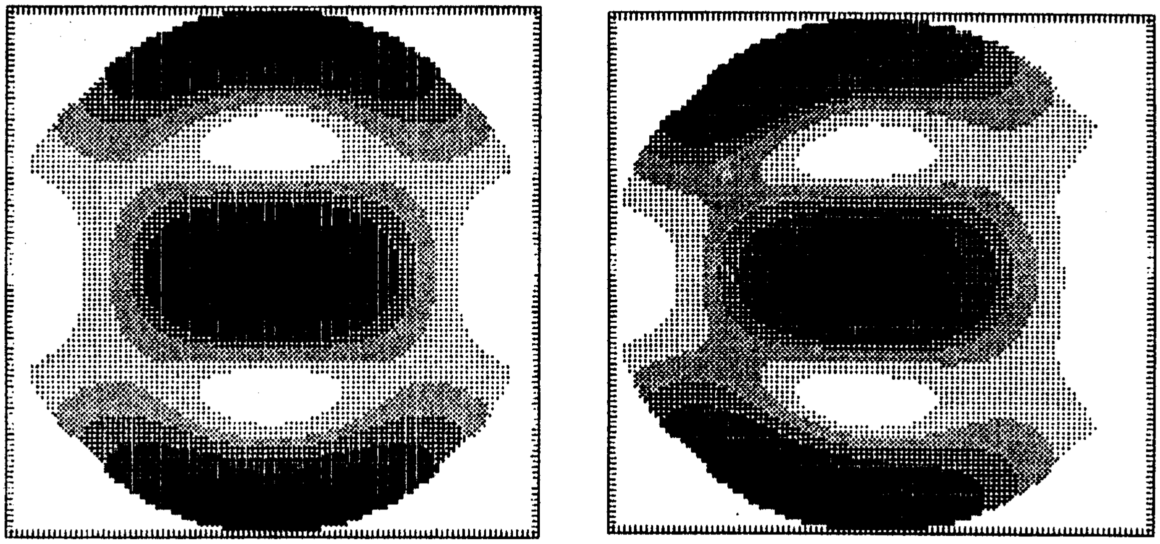

Figure 2. Energy flux density Transversal distribution of the longitudinal Poynting vector (see eq. (29)) for the TH_{11}-mode in the straight circular cylinder ($\delta = 0$) and its perturbed behaviour in the toroidal waveguide ($\delta = 0.042$).

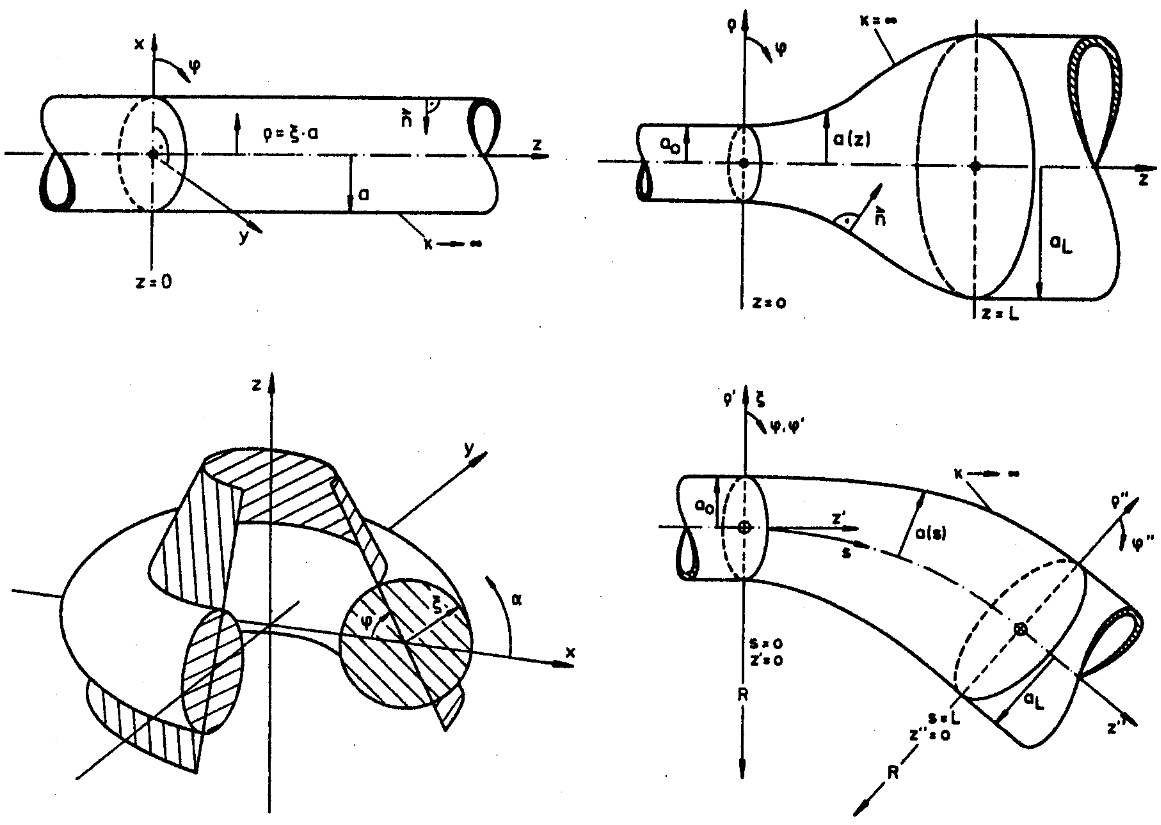

Figure 3. Hollow waveguides with circular cross section Circular cylinder, tapered waveguide, toroidal waveguide and tapered toroidal waveguide.